Taylor Methods and the Dynamical Analysis of Muon Accelerators

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Third International Workshop on Taylor Methods

17 December 2004

What is a Particle Accelerator?

- Guide charged particles along a particular path, using magnets
- That path is often circular, and the particles make many (10^8!) turns
- The particles are often accelerated from lower energies to higher energies
- Usually trying to do something with these particles eventually
  - Generally collide them into each other
  - Sometimes collide them into a stationary object
  - Sometimes make use of the radiation that they produce
  - For muons, sometimes we leave them in a ring and let them decay, producing a beam of neutrinos

Accelerator Hamiltonian

- Define reference curve $\boldsymbol{x}_0(s)$, s is arc length
- Coordinates are defined as deviations from this reference curve

$$\hat{s}(s) = \frac{d\boldsymbol{x}_0}{ds} \qquad \hat{x}(s) \cdot \hat{s}(s) = 0 \qquad \hat{y}(s) = \hat{s}(s) \times \hat{x}(s)$$
$$x = [\boldsymbol{z} - \boldsymbol{x}_0(s)] \cdot \hat{x} \qquad y = [\boldsymbol{z} - \boldsymbol{x}_0(s)] \cdot \hat{y}$$

- Hamiltonian dynamics
  - Independent variable is s
  - Hamiltonian, function of phase space coordinates (x, p_x, y, p_y, t, E)
  - Vector field based on derivatives of Hamiltonian: $(\partial_{p_x} H, -\partial_x H, \partial_{p_y} H, -\partial_y H, -\partial_E H, \partial_t H)$

Accelerator Hamiltonian (cont.)

$$
-(1+hx)A_s
- (1+hx)\sqrt{\left(\frac{E-q\phi}{c}\right)^2 - (p_x - qA_x)^2 - (p_y - qA_y)^2 - (mc)^2}
$$

- Simplified: ignoring torsion and vertical bending
- h is curvature of reference curve
- Electrostatic potential ϕ, magnetic vector potential (A_x, A_y, A_s)
  - Generally $\phi = 0$

Vector Potentials

- Vector potentials obey Maxwell's equations; for no curvature, time-independent magnetic fields:

$$A_s(x,y,s) = \sum_{m=1}\sum_{k=0}\frac{1}{k!(k+m)!}\Re\{A_m^{(2k)}(s)(x+iy)^m\}\left(-\frac{x^2+y^2}{4}\right)^k$$

$$A_x(x,y,s) = \sum_{m=1}\frac{1}{2m}\sum_{k=0}\frac{1}{k!(k+m)!}\Re\{A_m^{(2k+1)}(s)(x+iy)^m\}\left(-\frac{x^2+y^2}{4}\right)^k$$

$$A_y(x,y,s) = \sum_{m=1}\frac{1}{2m}\sum_{k=0}\frac{1}{k!(k+m)!}\Im\{A_m^{(2k+1)}(s)(x+iy)^m\}\left(-\frac{x^2+y^2}{4}\right)^k$$

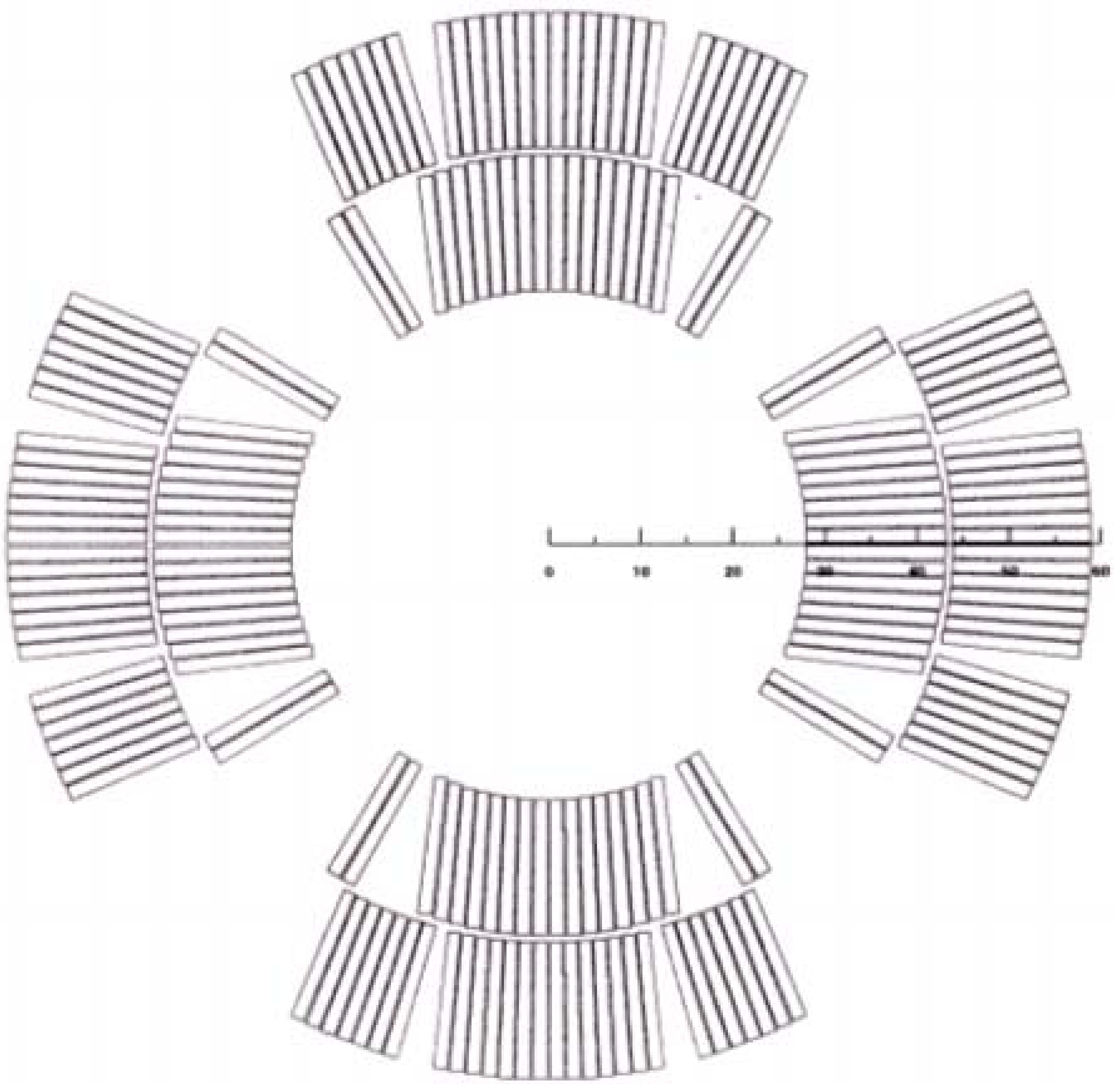
Basic Accelerator Design

- For given m, lowest order contribution to vector potential is

$$A_s(x, y, s) = \Re\{A_m(s)(x + iy)^m\}$$

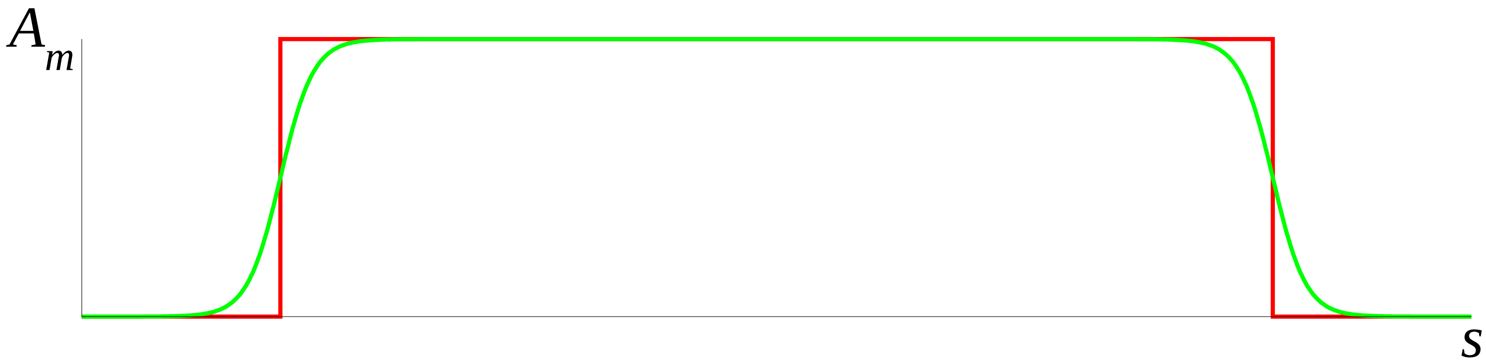
- When bending particle, use $A_s = -A_1(s)x$, with A_1 and h chosen together to keep particle on (near) reference curve (dipole)
  - I'm cheating, ignoring curvature terms here
- To give linear stability of orbits about reference curve, use term $A_s = A_2(s)(x^2 - y^2)/2$ (quadrupole)
  - Control other things we this also, discussed shortly
- Notice: these are low-order Taylor series, they generate low-order Taylor series in the Hamiltonian and vector field
  - If there were no s dependence, there would be no higher-order terms
- These A_s create linear terms in the vector field

Quadrupole Cross-Section



Longitudinal Field Dependence

- Field in magnet is designed to be as constant longitudinally as possible



- Short sections on ends where field is not constant, giving higher-order terms
- The integral of A_m gives the lowest-order effect
- The longer the magnet, the smaller the relative additional contribution of the ends is
  - End length is proportional to magnet aperture
  - All these effects generate higher-order terms
- A_x and A_y are higher-order

LHC Magnet Example

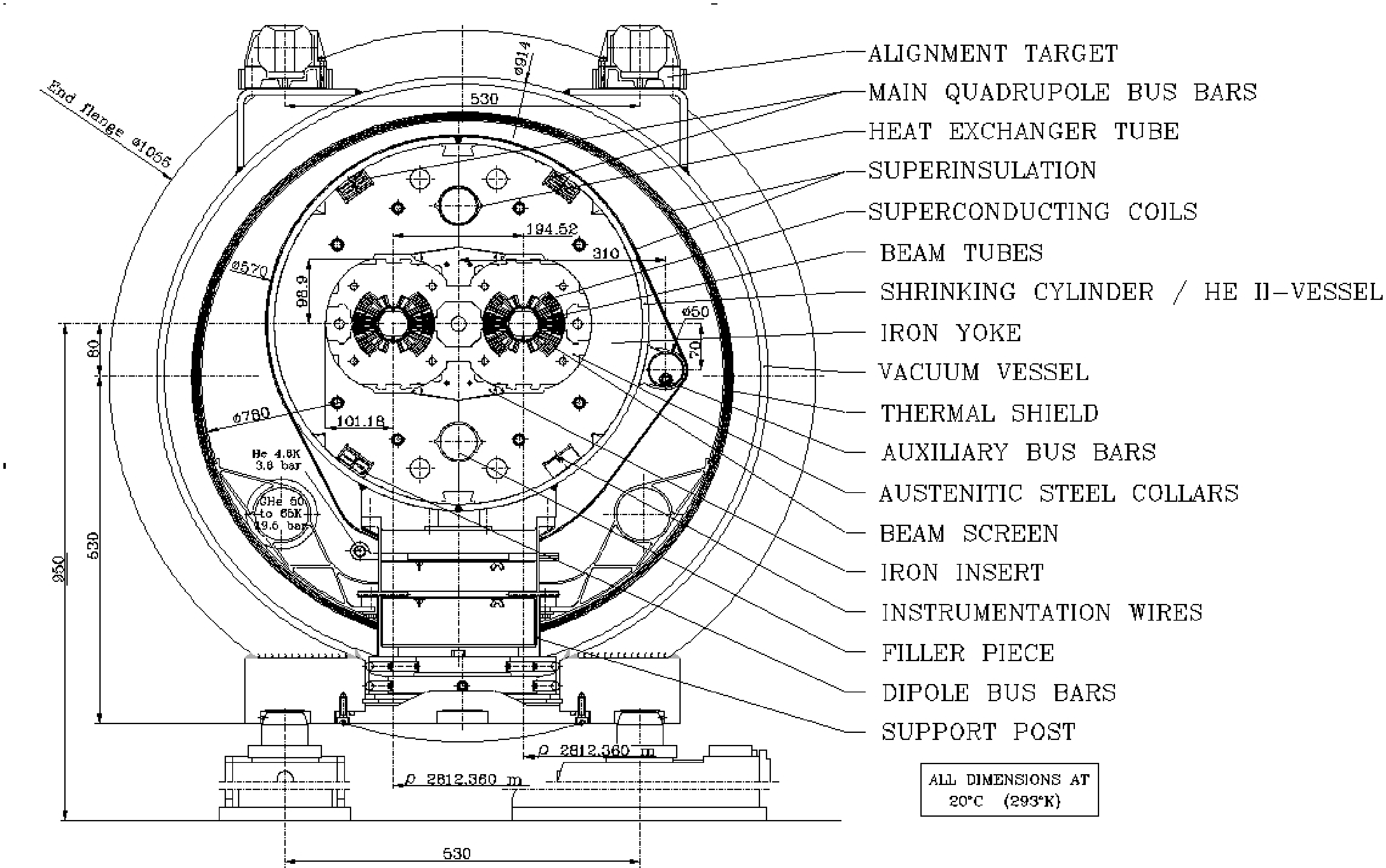
- Nominal dipole length is 14.312 m
- Inner coil diameter: 5.6 cm; outer coil diameter: about 11.8 cm.
- Thus, end length is small fraction of total length, leading to small relative importance of ends
- There are specific magnets for which ends are important, but they are small in number
- The RMS beam size in the arc is 1.7 mm
- There are higher multipole components in these magnets (A_m, with $m > 2$), which are kept to about 10^{-4} relative to the desired multipole component at a radius of 1.7 cm.
- Hamiltonian is linear in A_s

LHC Magnet Example (cont.)

- There are nonlinear magnets (A_m with $m > 2$) intentionally used, but they should give small contributions
- Conclusion:
  - Contributions of magnets to the vector field are primarily linear
  - Higher-order contributions are small
  - Magnetic contributions to map are well-represented by a rapidly converging Taylor series

LHC Magnet Photo

LHC Magnet Cross-Section



Angles

- Can expand square root part of Hamiltonian (ignore potentials, curvature) ($p = \sqrt{(E/c)^2 - (mc)^2}$

$$-p + \frac{p_x^2 + p_y^2}{2p} + \frac{(p_x^2 + p_y^2)^2}{8p^3} + \cdots$$

- Second term gives linear contribution to vector field
- Subsequent terms gives terms that are higher order in p_x/p and p_y/p
- LHC: the largest value for p_x/p is 1.2×10^{-4}
- Conclusion: square root contribution is well-represented by a rapidly converging Taylor series

Energy Spread

- Say we have a reference energy E_0; the energy is $E_0 + \Delta$
- Energy appears in above square root term as $\sqrt{(E/c)^2 - (mc)^2}$
- LHC: RMS $\Delta/pc = 3.06 \times 10^{-4}$
- Again, expect a rapidly converging Taylor series

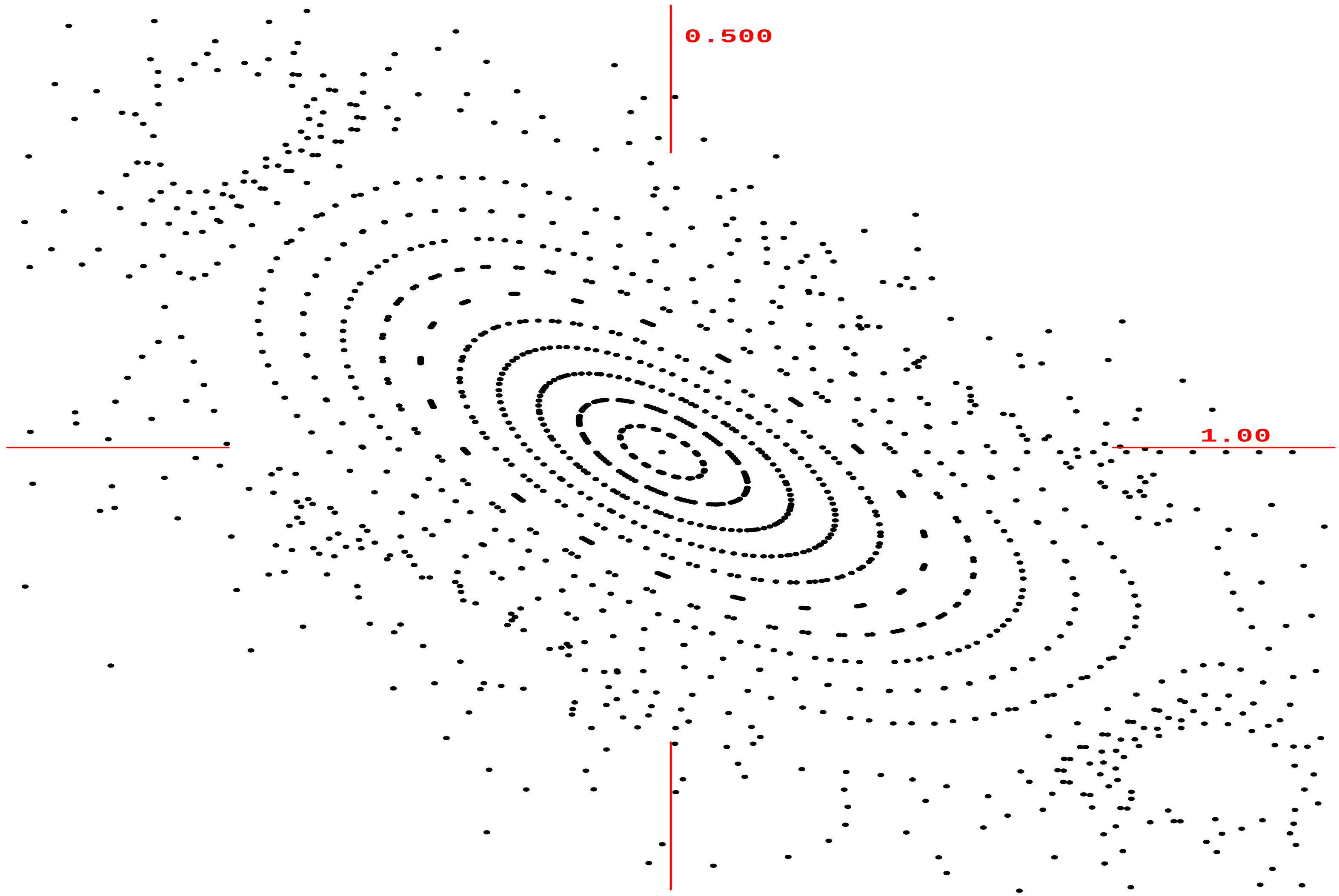
Curvature

- Terms like $1 + hx$ appear in Hamiltonian, h is curvature
- LHC: $h = 3.57 \times 10^{-4}\,\mathrm{m}^{-1}$, typical $x = 1.7$ mm
- Similar curvature terms appear in magnetic field expansions
- Again, terms are small, Taylor series converges rapidly

Conclusions for Standard Machines

- For most high energy particle accelerators
  - Vector field is well represented *globally* by a Taylor series
  - Terms in the Taylor series converge rapidly due to small amplitudes
  - Magnetic fields look like a Taylor series, by design
  - Linearized vector field gives a very good picture of the dynamics
- Cannot ignore nonlinear terms
  - With only linear terms, motion is stable for arbitrary displacements from the reference orbit
  - Due to small amplitudes, typically don't need an extremely high order Taylor series to represent the motion
  - Typically 8th order is sufficient for large machines
- Exceptions to this
  - Small rings: angles large, curvature large
  - Muon machines

Dynamic Aperture Plot



Applications of Taylor Methods

- Replace integration of equations of motion with evaluation of Taylor series
  - LHC contains around 2500 magnets
  - Beam makes around 6×10^8 turns
  - Integrating through each magnet can be prohibitive
  - Instead, construct Taylor series representing one-turn map
- Design an analysis: linear
  - Stability of fixed point
  - Beam sizes
- Design and analysis: nonlinear
  - Amplitude-dependent tunes (eigenvalues)
  - Resonances (higher-order fixed points away from reference orbit)

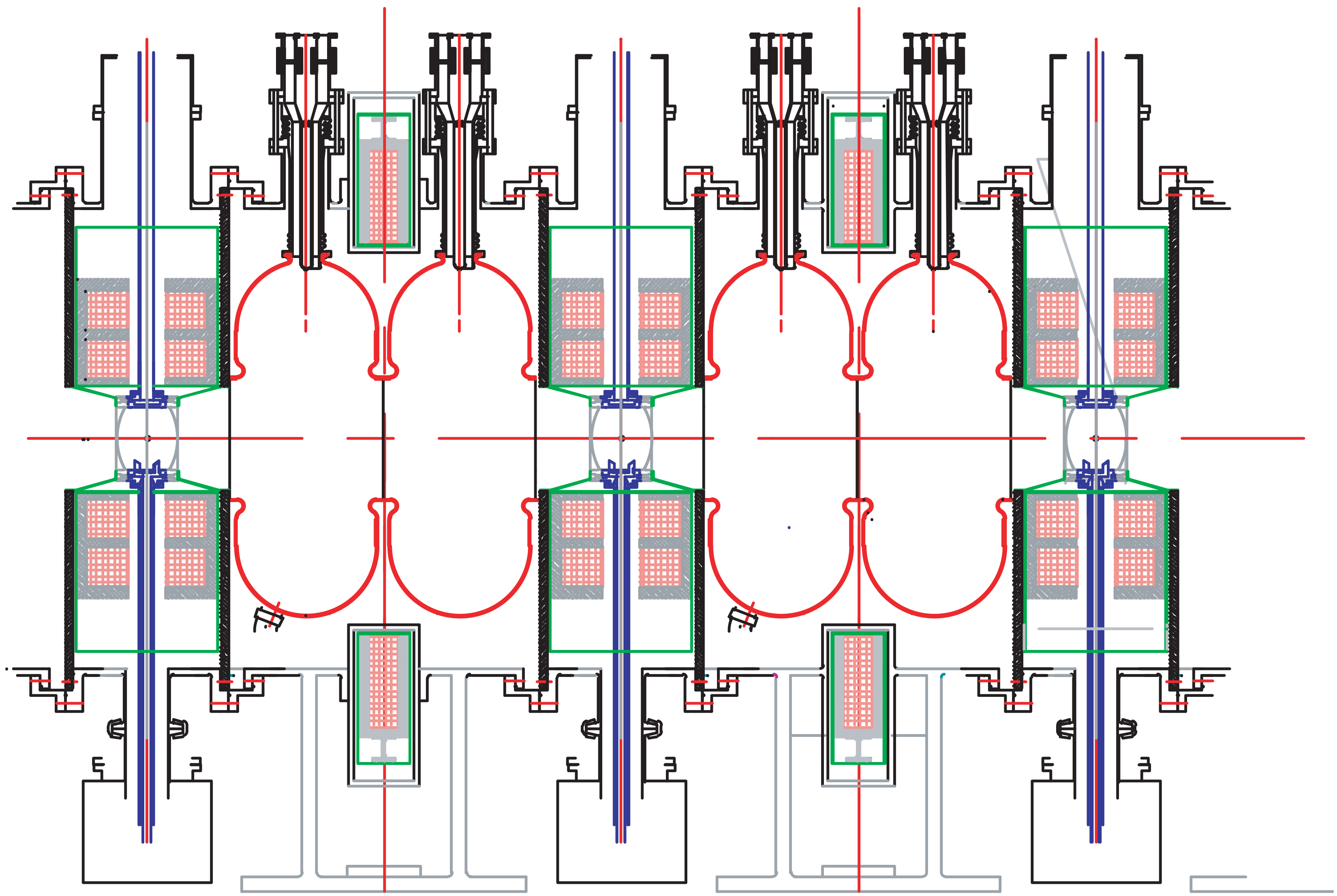
Muon Accelerators

- Beam sizes, energy spreads are large
- Different types of lattices
  - Cooling lattices
  - Fixed Field Alternating Gradient (FFAG) acceleration

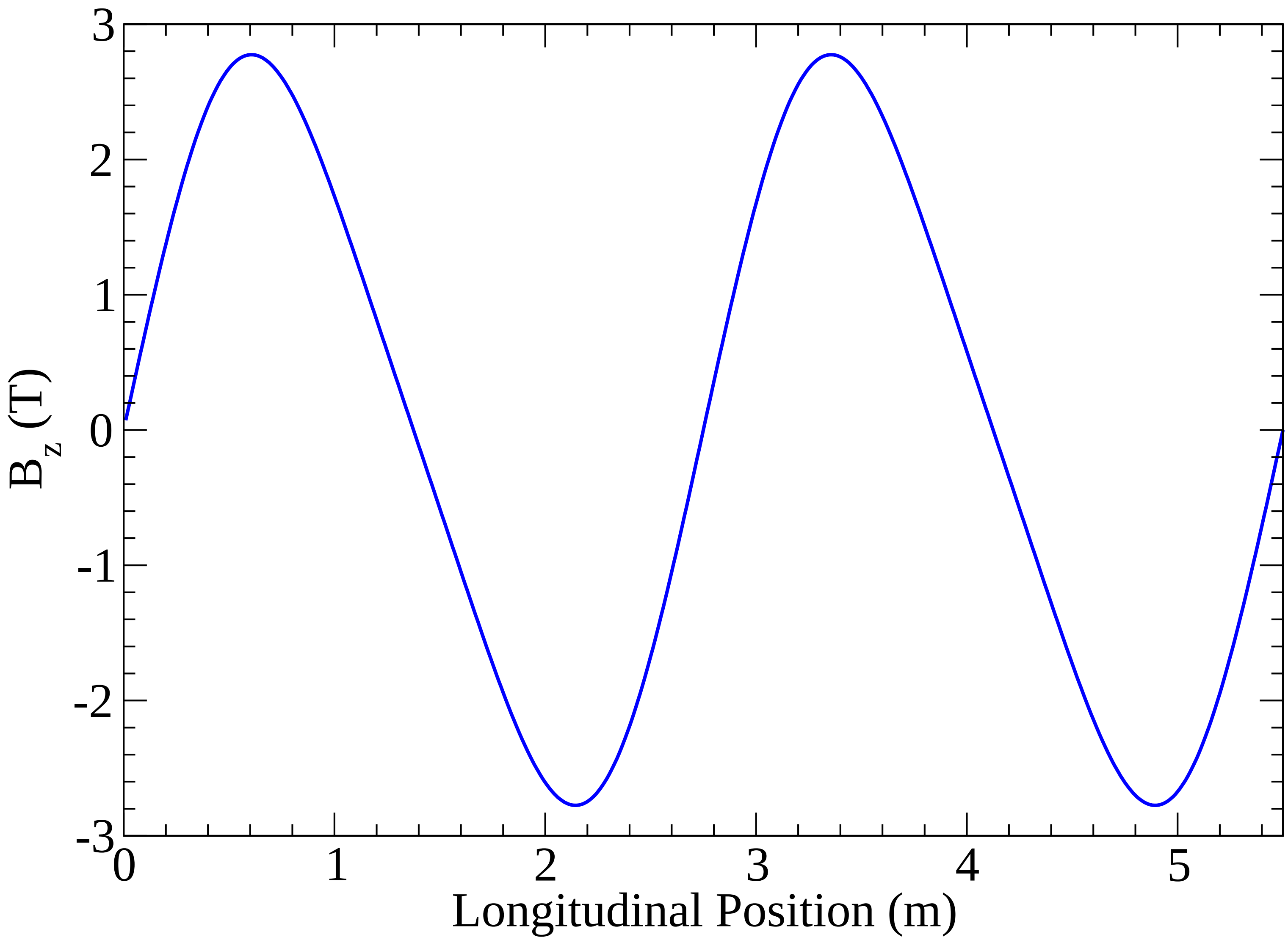
Lattice Types

- Cooling cells
  - Cooling requires extremely large angular spreads: as high as $p_x/p = 0.3$
  - Lattices may transport a factor of 2 in energy
  - Lattices must be compact to achieve these acceptances
  - Large beam size requires large magnet aperture
    - Beam fills significant fraction of aperture
  - Field no longer mostly constant
    - Higher order terms are significant
    - Series does not converge as quickly
- FFAG lattices
  - Normally when accelerating in a ring, increase magnetic fields with momentum: equations of motion (nearly) invariant
  - FFAG: keep fields fixed. Arc must accept a factor of 2 or more in energy

Cooling Cell



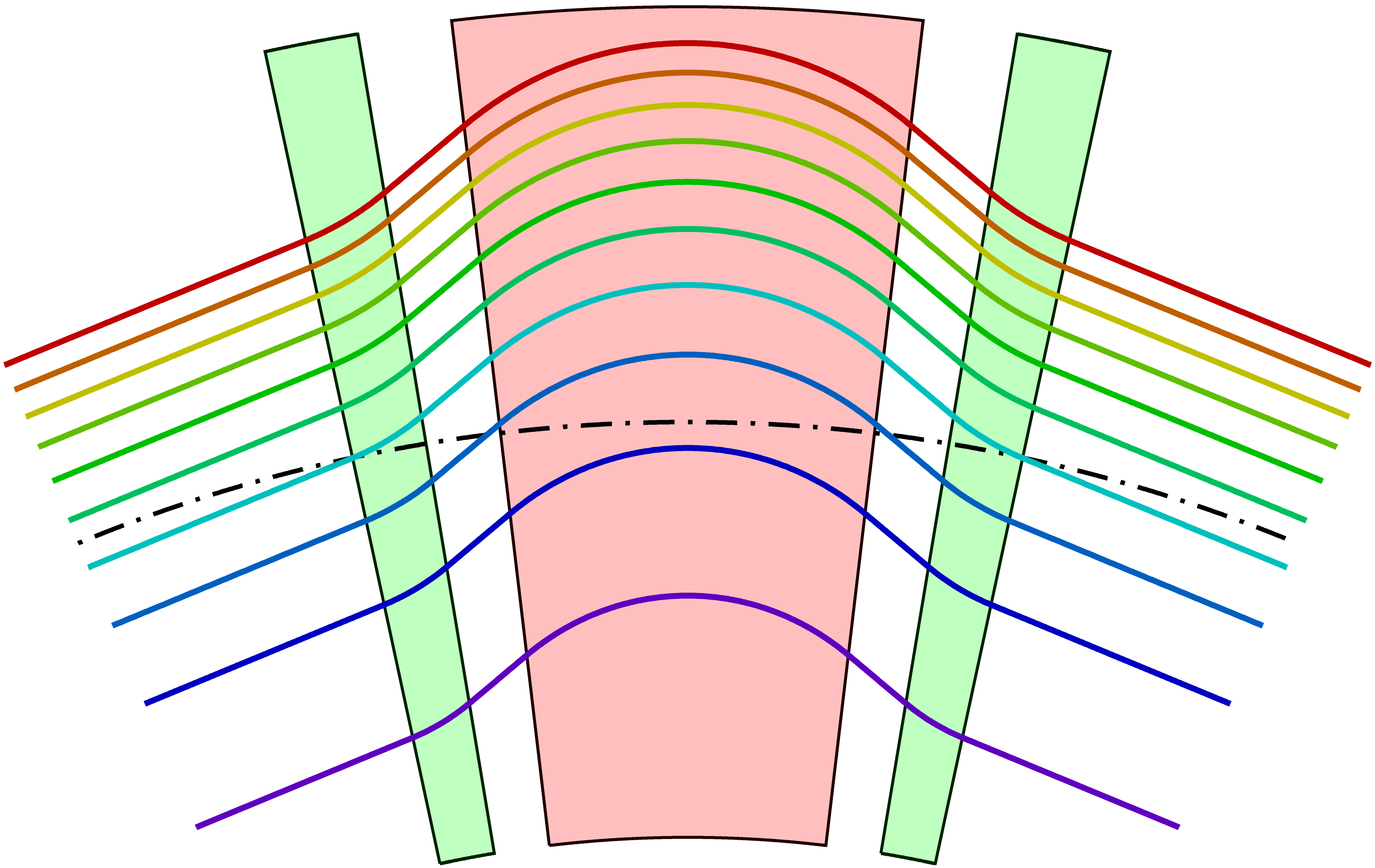
Field in Cooling Cell



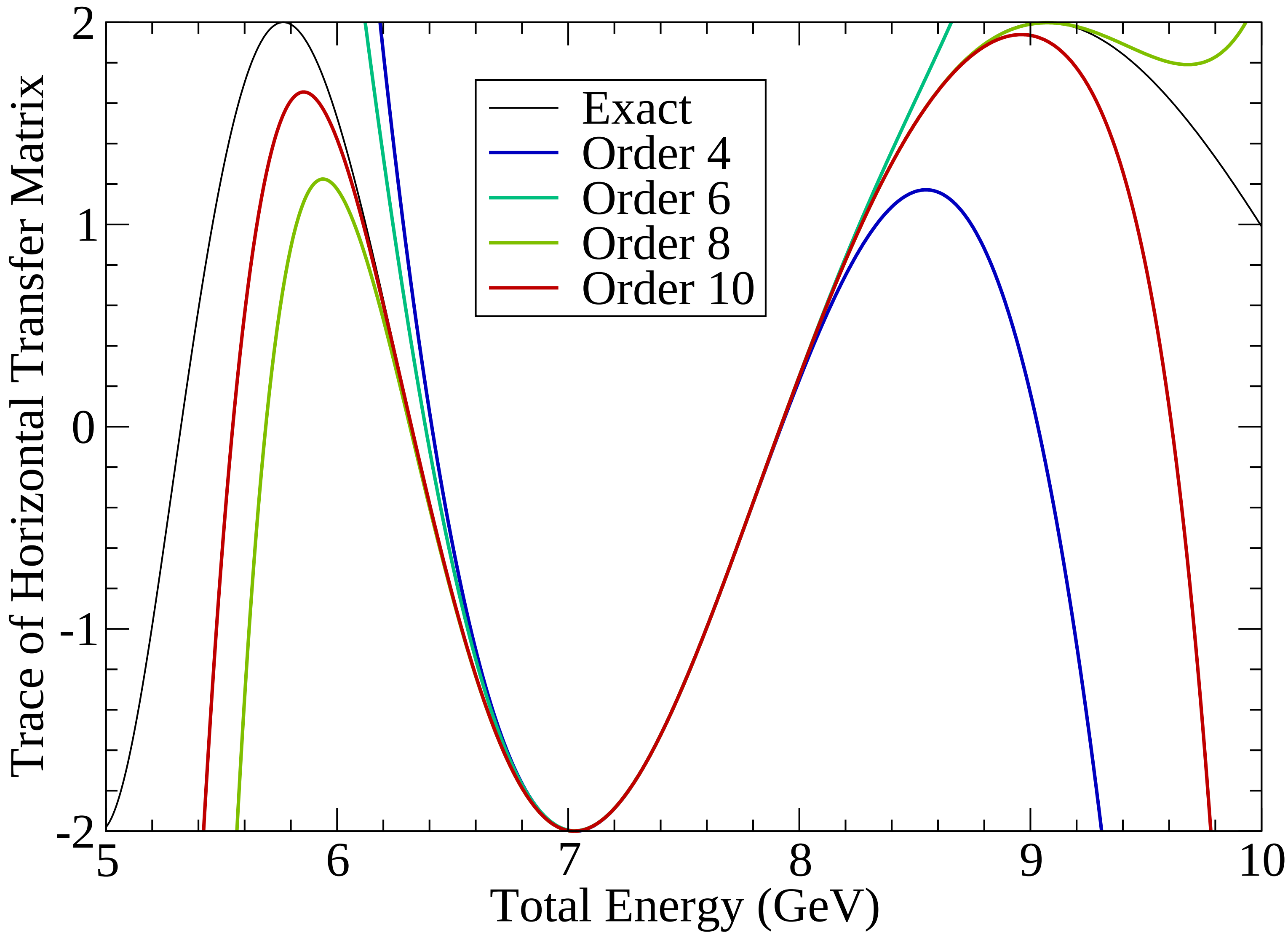
Wide Energy Range

- Analysis of lattice
  - Find fixed point of map for fixed Δ
  - The closed orbit is the phase space variables as a function of s passing through this fixed point
  - Translate coordinates to coordinates relative to this closed orbit
  - Write map in these new coordinates
  - Now have eigenvalues as a function of Δ, other linear lattice parameters
- Problem: truncation. Evalutaing Δ at large values gives effectively lower map in transverse variables
  - Linear lattice functions incorrect for large Δ
  - Tracking results manifestly wrong
- Solutions do not always converge with increasing order

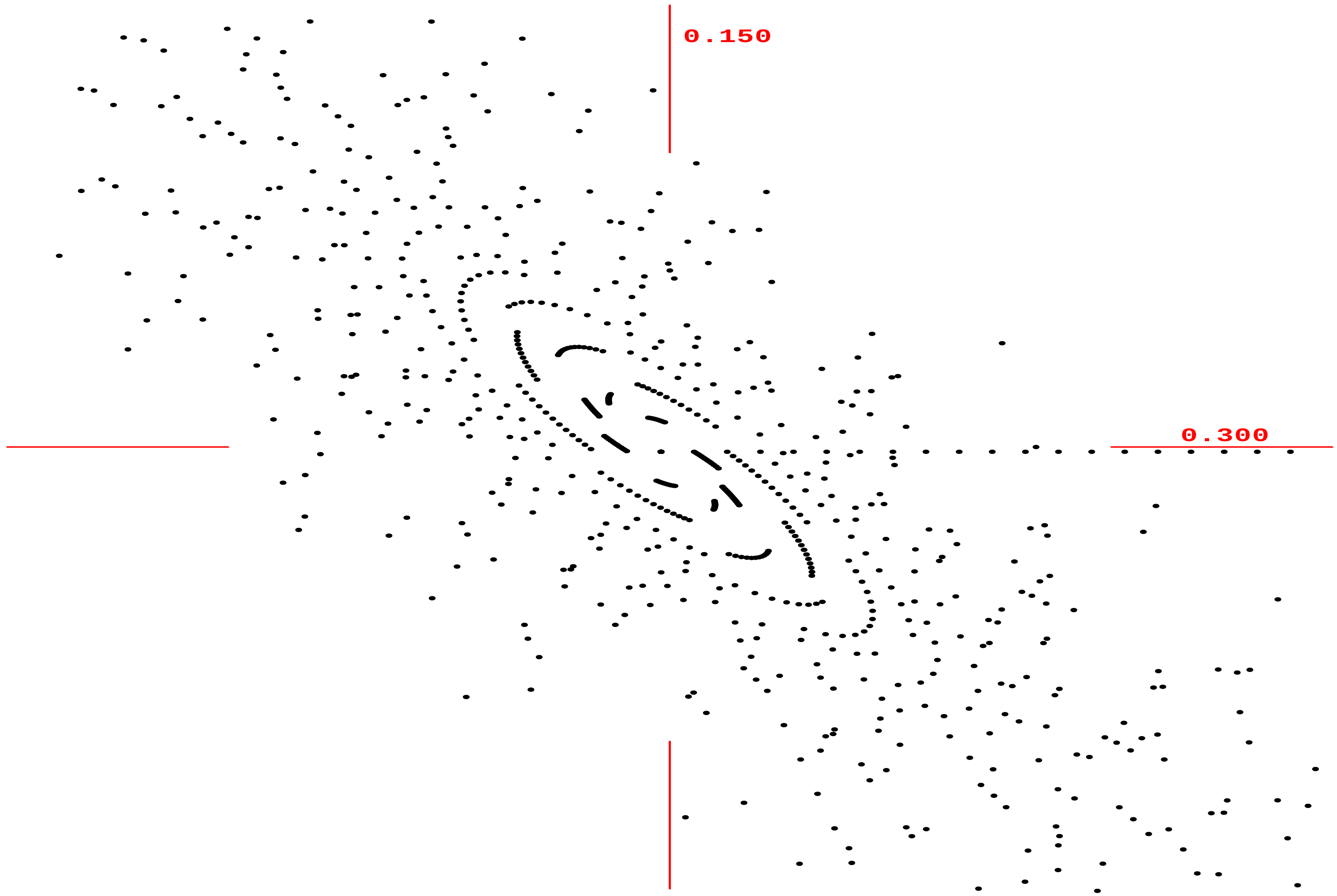
Closed Orbits



Matrix Trace Error



Off-Energy Tracking



Solutions for Wide Energy Range

- Need separate parameterization in Δ
  - Some kind of spline function
  - Don't make total order count order of polynomial in Δ
  - Domain decomposition
- Potentially have similar problems for large angles, highly nonlinear magnets, etc.

Conclusions

- Traditional accelerators seem to be designed to be represented by Taylor series
  - Idealized contribution of a magnet is a Taylor series
  - Most deviations are small, making the Taylor series converge quickly
  - A single Taylor series gives a global map representation
- In muon accelerators, deviations are not small
  - Leads to problems representing large energy ranges
    - To use Taylor methods, must parametrize energy differently
  - May also encounter problems resulting from
    - Large angles
    - Magnets whose end fields are significant